

On relaxing the constraints in pairwise compatibility graphs

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Abstract. A graph G is called a pairwise compatibility graph (PCG) if there exists an edge weighted tree T and two non-negative real numbers d_{min} and d_{max} such that each leaf l_u of T corresponds to a vertex $u \in V$ and there is an edge $(u, v) \in E$ if and only if $d_{min} \leq d_T(l_u, l_v) \leq d_{max}$ where $d_T(l_u, l_v)$ is the sum of the weights of the edges on the unique path from l_u to l_v in T . In this paper we analyze the class of PCG in relation with two particular subclasses resulting from the cases where $d_{min} = 0$ (LPG) and $d_{max} = +\infty$ (mLPG). In particular, we show that the union of LPG and mLPG does not coincide with the whole class PCG, their intersection is not empty, and that neither of the classes LPG and mLPG is contained in the other. Finally, as the graphs we deal with belong to the more general class of split matrogenic graphs, we focus on this class of graphs for which we try to establish the membership to the PCG class.

keywords: PCG, leaf power graph, threshold graphs, matrogenic graphs.

1 Introduction

Given an edge weighted tree T , let d_{min} and d_{max} be two nonnegative real numbers with $d_{min} \leq d_{max}$. For any two leaves l_1 and l_2 of the tree T , we denote by $d_T(l_1, l_2)$ the sum of the weights of the edges on the unique path from l_1 to l_2 in T . Starting from T , d_{min} and d_{max} , it can be easily constructed a *pairwise compatibility graph* of T , i.e. a graph $G(V, E)$ where each vertex $u \in V$ corresponds to a leaf l_u of T and there is an edge $(u, v) \in E$ if and only if $d_{min} \leq d_T(l_u, l_v) \leq d_{max}$. We will denote such a graph G by $PCG(T, d_{min}, d_{max})$. Consequently, we say that a graph G is a pairwise compatibility graph (PCG) if there exists an edge weighted tree T and two nonnegative real numbers d_{min} and d_{max} such that $G = PCG(T, d_{min}, d_{max})$. Determine whether a graph G is a PCG seems in general difficult even if at the beginning, when the problem arose in a computational biology context [2], it was conjectured that every graph was a PCG. Nowadays it is known that this conjecture is false [5], while it is proved that some specific classes of graphs e.g., graphs with five nodes or less [4], cliques and

disjoint union of cliques [1], chordless cycles and single chord cycles [6] and some particular subclasses of bipartite graphs [5], are PCG.

The pairwise compatibility concept is defined with respect to two bounds concerning d_{min} and d_{max} . If we relax these conditions, requiring only that the distance between some pair of leaves is smaller than or equal to d_{max} (i.e. we set $d_{min} = 0$) then we are considering a particular subclass of PCG graphs, namely the *leaf power* graphs (LPG). More formally, a graph $G(V, E)$ is a leaf power if there exists a tree T and a nonnegative number d_{max} such that there is an edge (u, v) in E if and only if for their corresponding leaves l_u, l_v we have $d_T(l_u, l_v) \leq d_{max}$ (see [3]). Although there has been a lot of works on this class of graphs [1], a completely description of leaf power graphs is still unknown.

To the best of our knowledge, nothing is known in literature concerning the subclass of PCG when the constraint concerns only the minimum distance, i.e. there is an edge in E if and only if the corresponding leaves are at a distances greater than k in the tree (observe that in this case we set $d_{max} = +\infty$). In this paper we introduce this new concept and exploit the relations between the new defined class and the two known classes LPG and PCG.

The paper is organized as follows: in Section 2 we introduce some terminologies and recall some known concepts that we will use in the forthcoming work. Then, we define the new subclass of PCG, namely mLPG, characterized by the use of d_{min} only. Next, in Section 3 we study the relations between the classes PCG, LPG and mLPG. In particular, we show that the union of LPG and mLPG does not coincide with the whole class PCG, their intersection is not empty, and neither of the classes LPG and mLPG is contained in the other. All the graphs we furnish as examples in Section 3 are particular cases of the more general class of split matrogenic graphs. Hence, in Section 4 we focus on the class of split matrogenic graphs trying to determine if it belongs to the PCG class. We prove that many split matrogenic graphs are PCG. However, the membership to PCG class of one particular subclass of split matrogenic graph remains an open problem that is reported in the final Section 5 together with some other open problems.

2 Preliminaries

In this section we introduce some definitions and some concepts that we use in the rest of this paper.

When we say that a *tree* T is *weighted*, we mean that it is edge weighted, that is each edge is assigned a number as its weight. In this paper we consider only weighted trees and graphs that are connected.

A *caterpillar* is a tree in which all the vertices are within distance one of a central path which is called the *spine*.

A graph $G = (K, S, E)$ is said to be *split* if there is a vertex partition $V = K \cup S$ such that the subgraphs induced by K and S are complete and stable, respectively.

Given two split graphs $G_1 = (K_1, S_1, E_1)$ and $G_2 = (K_2, S_2, E_2)$ their *composition* $G_1 \circ G_2$ is formed by taking the disjoint union of G_1 and G_2 and adding all the edges $\{u, v\}$ such that $u \in K_1$ and $v \in V(G_2)$. Observe that $G_1 \circ G_2$ is again a split graph.

A set M of edges is a *perfect matching* of dimension m of A onto B if and only if A and B are disjoint subsets of vertices of cardinality m and each vertex in A is adjacent to exactly one vertex in B and no two edges share a point. We say that the split graph $G = (K, S, E)$ is a *split matching* if the subset of edges in E not belonging to the clique forms a perfect matching. We denote by \mathcal{SM} the class of split matching graphs.

An *antimatching* of dimension m of A onto B is a set of edges such that the non edges between A and B form a perfect matching. We say that the split graph $G = (K, S, E)$ is a *split antimatching* if the subset of edges in E not belonging to the clique forms an antimatching. We denote by \mathcal{SA} the class of split antimatching graphs.

A *cactus* is a connected graph in which any two simple cycles have at most one vertex in common. Equivalently, every edge in such a graph may belong to at most one cycle. We will denote by \mathcal{C} the class of cacti with at least a cycle of length $n \geq 5$.

Given a connected graph G whose distinct vertex degrees are $\delta_1 > \dots > \delta_r$, we define $B_i = \{v \in V(G) : \deg(v) = \delta_i\}$, for any $i = 1, \dots, r$. The sets B_i are usually referred as *boxes* and the sequence B_1, \dots, B_r is called the *degree partition of G into boxes*.

Given a graph G with degree partition B_1, \dots, B_r , G is a *threshold graph* if and only if for all $u \in B_i$, $v \in B_j$, $u \neq v$, we have $(u, v) \in E(G)$ if and only if $i + j \leq r + 1$. We will denote by \mathcal{T} the class of threshold graphs.

Now, we introduce a new subclass of PCG, namely mLPG as follows:

Definition 1 *A graph $G(V, E)$ is an mLPG if there exists a tree T and an integer d_{min} such that there is an edge (u, v) in E if and only if for their corresponding leaves l_u, l_v in T we have $d_T(l_u, l_v) \geq d_{min}$.*

Note that for the sake of simplicity and homogeneity of the paper, here we slightly abuse notation as these graphs are not power of trees.

In what follows we will often make use of the following simple observation.

Proposition 1 *Let G be a graph that does not belong to some class L from $\{PCG, LPG, mLPG\}$ then every graph H that contains G as an induced subgraph, does not belong to L either.*

Given two vertices u, v in a tree T , we denote by P_{uv} the unique path in T connecting the vertices u and v . A *subtree induced by a set of leaves* of T is the minimal subtree of T which contains those leaves. We denote by T_{uvw} the subtree of a tree induced by three leaves u, v and w .

The following technical lemma will turn out to be very useful in the forthcoming results.

Lemma 1 [5] *Let T be an edge weighted tree, and u, v and w be three leaves of T such that P_{uv} is the largest path in T_{uvw} . Let x be a leaf of T other than u, v and w . Then, either $d_T(w, x) \leq d_T(u, x)$ or $d_T(w, x) \leq d_T(v, x)$.*

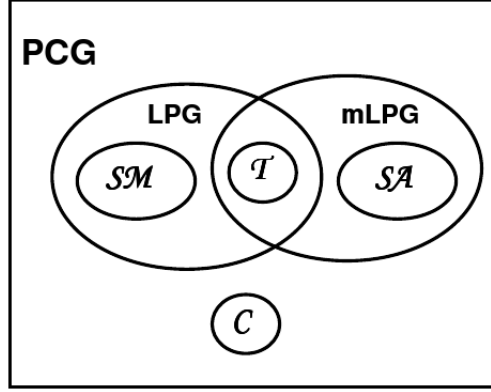


Fig. 1: Relationships between PCG, LPG and mLPG.

3 Relationships between PCG, LPG and mLPG

In this section we study the relationships between the classes of PCG , LPG and $mLPG$. First, in Subsection 3.1 we show that the union of $mLPG$ and LPG does not contain the whole class of PCG . Next, in Subsection 3.2 we show that their intersection $LPG \cap mLPG$ is not empty, by proving that threshold graphs belong to both classes. Finally, in Subsection 3.3 we show that neither of the classes LPG and $mLPG$ is contained in the other one by providing for each of these classes a particular graph which is proper to it. These relations are graphically shown in Figure 1.

3.1 $PCG \supset LPG \cup mLPG$

In this subsection we prove that the PCG class does not coincide with the union of LPG and $mLPG$. Indeed, in [6] it is proved that any cycle is a PCG. Now, it is well-known (see, for example, [1]) that LPG is a subclass of strongly chordal graphs and clearly cycles of length $n \geq 5$ are not strongly chordal, so they are not LPG . The following lemma states that cycles do not belong to $mLPG$, deducing that $(LPG \cup mLPG) \subset PCG$.

Lemma 2 *Let C_n be a cycle of length $n \geq 5$, then $C_n \notin mLPG$.*

Proof. Let v_1, \dots, v_n be the ordered vertices of a cycle C_n with $n \geq 5$. Suppose by contradiction that $C_n = \text{mLPG}(T, d_{\min})$ and let l_i be the leaf in T corresponding to the vertex v_i , for any $i \leq n$. Let v_1, v_2, v_3 be the first three consecutive vertices in C_n and consider the largest path in $T_{l_1 l_2 l_3}$. As $(v_1, v_3) \notin E$ (as $n \geq 5$) then $d_T(l_1, l_3) < d_{\min}$. Hence, the largest path must be one from $P_{l_1 l_2}$ and $P_{l_2 l_3}$.

Suppose first the largest path is $P_{l_1 l_2}$. Using Lemma 1 with $x = l_4$ we have that either $d_{\min} \leq d_T(l_4, l_3) \leq d_T(l_4, l_2)$ or $d_{\min} \leq d_T(l_4, l_3) \leq d_T(l_4, l_1)$, deducing that at least one between the (v_4, v_2) and (v_4, v_1) must be an edge in C_n , a contradiction.

If $P_{l_2 l_3}$ is the largest path, we arrive at the same result by taking this time $x = l_n$. This concludes the proof. \square

Easily, in view of Proposition 1 the class \mathcal{C} of cacti with at least one cycle of length $n \geq 5$ does not belong either to LPG or to mLPG.

3.2 $\text{LPG} \cap \text{mLPG} \neq \emptyset$

In this subsection we prove that the intersection of LPG and mLPG is not empty by showing that threshold graphs belong to $\text{LPG} \cap \text{mLPG}$.

Theorem 1 *Let G be a threshold graph. Then $G \in \text{LPG} \cap \text{mLPG}$ and it is polynomial to find the tree T and the values d_{\min}, d_{\max} associated to G .*

Proof. Let G be a threshold graph on n vertices and let B_1, \dots, B_r be the degree partition of G . We consider an n -leaf star with center a vertex c , as the tree T .

To prove that $G \in \text{LPG}$, for each vertex v of G , assign weight i to the edge (l_v, c) in T if $v \in B_i$. Define $d_{\max} = r + 1$. As for each $u \in B_i, v \in B_j, u \neq v$, we have $(u, v) \in E(G)$ if and only if $i + j \leq r + 1$ it is straightforward that G is a PCG of T with d_{\max} .

On the other hand, to prove $G \in \text{mLPG}$ for any $v \in V(G)$ assign $r + 1 - i$ to the edge (l_v, c) in T if $v \in B_i$. Note that, as $i \leq r$ we assign nonnegative weights to the edges of the star. Define $d_{\min} = r + 1$. For any two vertices $v \in B_i$ and $u \in B_j$, we have that if $i + j \leq r + 1$ (meaning that $(u, v) \in E(G)$) then $d_T(l_u, l_v) = 2(r + 1) - (i + j) \geq r + 1 = d_{\min}$. Otherwise, if $i + j > r + 1$ (meaning that $(u, v) \notin E(G)$) then $d_T(l_u, l_v) = 2(r + 1) - (i + j) < r + 1 = d_{\min}$. This concludes the proof. \square

3.3 $\text{LPG} \setminus \text{mLPG} \neq \emptyset$ and $\text{mLPG} \setminus \text{LPG} \neq \emptyset$

Here we show that neither of the classes LPG and mLPG is contained in the other one by providing, for each of these classes a particular graph which is proper to it.

Theorem 2 *Let G be a split matching graph. Then $G \notin \text{mLPG}$, $G \in \text{LPG}$ and in this case it is polynomial to find the tree T and the value d_{\max} associated to G .*

The proof will follow immediately by the next two lemmas.

Lemma 3 *Let G be a split matching graph. Then $G \in LPG$ and it is polynomial to find the tree T and the value d_{max} associated to G .*

Proof. Given a split matching graph $G = (K, S, E)$ with $|K| = |S| = n$, we associate a caterpillar tree T as in Figure 2(a). The leaves a_i , corresponding to the vertices k_i of K , are connected to the spine with edges of weight 1 and the leaves b_i , corresponding to vertices s_i of S , with edges of weight n . It is clear that $G = LPG(T, n+1)$. Indeed, for any two a_i, a_j it holds that $3 \leq d_T(a_i, a_j) \leq n+1$, for any two b_i, b_j we have $d_T(b_i, b_j) \geq 2n+1$ and for any a_i, b_i we have $d_T(a_i, b_i) = n+1$ (hence the edge $(k_i, s_i) \in E$) and for any a_i, b_j with $i \neq j$ we have $d_T(a_i, b_j) \geq n+2$ (hence the edge $(k_i, s_j) \notin E$). \square

Note that this representation is not unique. Indeed, one can easily check that the binary tree T in Figure 2(b) also is a pairwise compatibility tree of a split matching graph when $d_{max} = 4$.

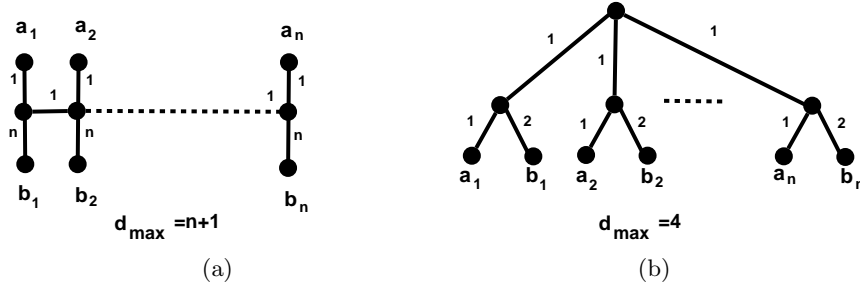


Fig. 2: (a) A pairwise compatibility caterpillar tree for a split matching graph. (b) A pairwise compatibility tree for a split matching graph.

Lemma 4 *Let G be a split matching graph. Then $G \notin mLPG$.*

Proof. Given a split matching graph $G = (K, S, E)$ with $|K| = |S| = n$, we assume by contradiction $G = mLPG(T, d_{min})$. Then let a_1, a_2, a_3 be three leaves of T corresponding to three vertices of K , k_1, k_2, k_3 . Without loss of generality let $P_{a_1 a_2}$ be the largest path in the subtree $T_{a_1 a_2 a_3}$. Consider the vertex s_3 in S associated to the leaf b_3 in T , with $(k_3, s_3) \in E$. From Lemma 1 we deduce that either $d_T(b_3, a_3) \leq d_T(b_3, a_2)$ or $d_T(b_3, a_3) \leq d_T(b_3, a_1)$. The existence of the edge (k_3, s_3) in G implies $d_T(b_3, a_3) \geq d_{min}$, therefore one from (k_1, s_3) and (k_2, s_3) must be an edge in G , a contradiction. \square

Analogously, we can show that the set $mLPG \setminus LPG$ is not empty.

Theorem 3 *Let G be a split antimatching graph. Then $G \notin LPG$, $G \in mLPG$ and in this case it is polynomial to find the tree T and the value d_{min} associated to G .*

For the sake of brevity we omit the proof of this theorem, that follows using arguments similar to those in the proofs of Lemmas 3 and 4. The tree T associated to a split antimatching graph is one from the ones depicted in Figure 3.

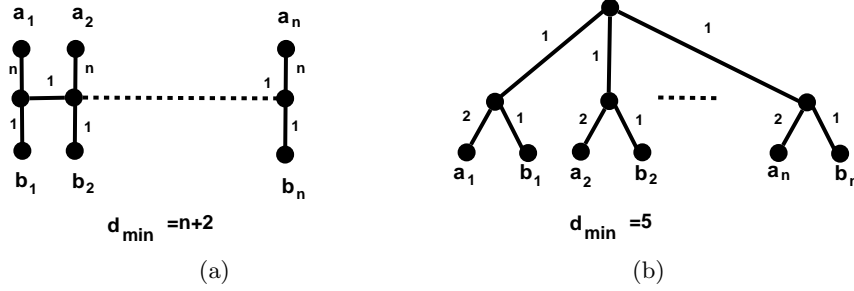


Fig. 3: (a) A pairwise compatibility caterpillar tree for a split antimatching graph. (b) A pairwise compatibility tree for a split antimatching graph.

4 Split Matrogenic Graphs

In Section 3, when studying the relations among the three classes PCG, LPG and mLPG, we have dealt with threshold graphs, split matchings and split antimatchings. All these graphs are split matrogenic graphs (cfr. definition later). For this reason, it is natural to ask whether split matrogenic graphs are PCG or not. This section is devoted to answer this question.

Definition 2 A split matrogenic graph is the composition of t split graphs $G_i = (K_i, S_i, E_i)$ with $i = 1, \dots, t$ such that: either G_i is a split matching or G_i is a split antimatching or $K_i = \emptyset$ (and G_i is called stable graph) or $S_i = \emptyset$ (and G_i is called clique graph).

In order to make easier the exposition, we introduce two subclasses of split matrogenic graphs.

Definition 3 Given a sequence of t split graphs $G_i = (K_i, S_i, E_i)$ with $i = 1, \dots, t$, we say the graph $H = G_1 \circ \dots \circ G_t$ is a split matching (antimatching) sequence if each of the graphs G_i is either a split matching (antimatching), or a stable graph or a clique graph.

We first prove that split matching sequences and split antimatching sequences are PCG. In both these proofs, in the construction of the pairwise compatibility tree, we will make use of the constructions depicted in Figure 2(b) and Figure 3(b), respectively. Finally, we want to point out that a clique graph (a stable

graph) can be considered both as a split matching graph or a split antimatching graph and in each case the pairwise compatibility tree is constructed in the same way, where only leaves a_i (respectively b_i) appear. In Figure 4 the pairwise compatibility tree is given for a stable graph G when it is considered first as a split matching graph and next as a split antimatching graph.

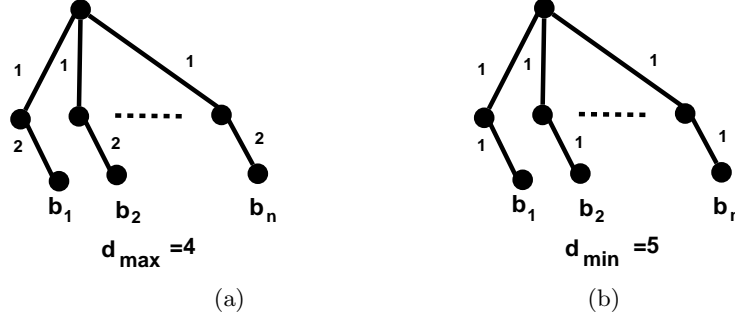


Fig. 4: The pairwise compatibility tree for a stable graph G with n vertices when it is considered as (a) a split matching graph (b) a split antimatching graph.

Theorem 4 *Let H be a split matching sequence. Then $H \in LPG$ and it is polynomial to find the tree and the value d_{\max} associated to H .*

Proof. Let $H = G_1 \circ \dots \circ G_t$ be a split matching sequence. For each graph G_i we define a tree T_i as shown in Figure 5(a) (where the leaves a_i (b_i) may not possibly appear if G_i is a stable (clique) graph). It is clear that $G_i = LPG(T_i, d_{\max})$ where d_{\max} is a value to be defined later, but surely greater than or equal to $2(i+1)$. Indeed, let a_1, \dots, a_n be the leaves of T_i corresponding to vertices of K_i and b_1, \dots, b_n those corresponding to vertices of S_i . For any two leaves a_r, a_s it holds that $d_{T_i}(a_r, a_s) = 2 + 2i \leq d_{\max}$ and for any two b_s, b_r we have $d_{T_i}(b_r, b_s) = 2d_{\max} - 2i \geq d_{\max} + 2i + 2 - 2i > d_{\max}$. Finally, for any two leaves a_s, b_s that correspond to an edge of the matching their distance is $d_{\max} - 2i + 1 \leq d_{\max}$ and for any two leaves corresponding to a non edge a_r, b_s their distance is $d_{\max} + 1$.

In order to prove that $H \in LPG$, we define a new tree T starting from the trees T_1, \dots, T_t , simply by contracting all their roots to a single vertex as shown in Figure 5(b). We claim that $H = LPG(T, d_{\max})$ where we set $d_{\max} = 2(t+1)$. In order to prove it, consider two graphs G_i and G_j with $i < j$. Let a, a', b and b' be four distinct leaves corresponding to vertices in K_i, K_j, S_i and S_j respectively. Observe that the vertices in K_i are connected to all the other vertices in $K_j \cup S_j$ as the distances in T are $d_T(a, a') = 1 + i + j + 1 \leq 2(j+1) \leq d_{\max}$ and $d_T(a, b') = 1 + i + j + d_{\max} - 2j = d_{\max} + (i - j + 1) \leq d_{\max}$ (as $j \geq i + 1$). Finally, any vertex in S_i is not connected to any vertex K_j and to any vertex S_j

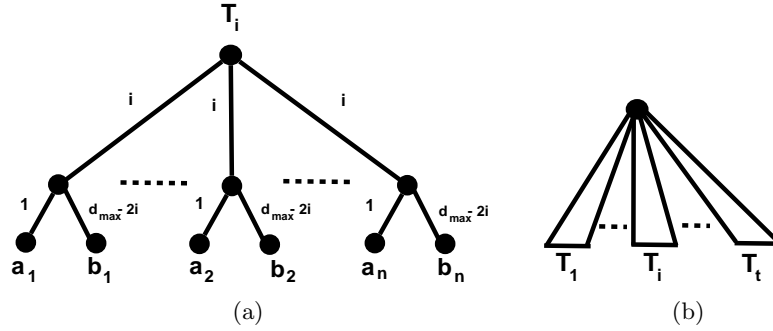


Fig. 5: (a) The pairwise compatibility tree for the split matching graph G_i . (b) The pairwise compatibility tree for the split matching sequence H .

as in these cases the distances are $d_T(b, a') = d_{\max} - 2i + i + j + 1 > d_{\max}$ (as $j \geq i + 1$) and $d_T(b, b') = d_{\max} - 2i + i + j + d_{\max} - 2j \geq 2d_{\max} - 2j > d_{\max}$. \square Using similar arguments we prove the following result.

Theorem 5 *Let H be a split antimatching sequence. Then $H \in mPCG$ and it is polynomial to find the tree and the value d_{\min} associated to H .*

Proof. The proof follows the same lines of the proof of Theorem 4. Let $H = G_1 \circ \dots \circ G_t$ be a split antimatching sequence. We will associate to each split antimatching graph G_i a tree T_i as depicted in Figure 6. We prove that $G_i = mLPG(T_i, d_{\min})$ where d_{\min} is a value to be defined later, but surely greater than or equal to $2(i+1)+1$. Indeed, let a_1, \dots, a_n be the leaves of T_i corresponding to vertices of K_i and b_1, \dots, b_n those corresponding to vertices of S_i . For any two leaves a_r, a_s it holds that $d_{T_i}(a_r, a_s) = 2d_{\min} - 2i - 2 = d_{\min} + (\dim - 2(i+1)) \geq d_{\min}$ and for any two b_s, b_r we have $d_{T_i}(b_r, b_s) = 2i + 2 < d_{\min}$. Finally, for any two leaves a_s, b_s that correspond to an edge of the antimatching their distance is $d_{\min} - 2i - 1 + 2i + 1 = d_{\min}$ and for any two leaves corresponding to a non edge a_r, b_r their distance is $d_{\min} - 2i$.

We define the tree T starting from the trees T_1, \dots, T_t , in the same way we have done in the previous theorem (see Figure 5(b)). Using the same arguments it is not difficult to check that $H = mLPG(T, d_{\min})$ where we have set $d_{\min} = 2(t+1)+1$. Indeed, consider two graphs G_i and G_j with $i < j$. Let a, a', b and b' be four distinct leaves corresponding to vertices in K_i, K_j, S_i and S_j respectively. Observe that the vertices in K_i are connected to all the other vertices in $K_j \cup S_j$ as the distances in T are $d_T(a, a') = d_{\min} - i - 1 + d_{\min} - j - 1 = \dim + (d_{\min} - (i + j + 2)) \geq d_{\min}$ and $d_T(a, b') = d_{\min} - i - 1 + j + 1 \geq d_{\min}$. Finally, any vertex in S_i is not connected to any vertex K_j and to any vertex S_j as in these cases the distances are $d_T(b, a') = 1 + i + j + d_{\min} - 2j - 1 = d_{\min} + i - j < d_{\min}$ (as $j \geq i + 1$) and $d_T(b, b') = 1 + i + j + 1 < 2(j + 1) \leq 2(t + 1) < d_{\min}$. \square

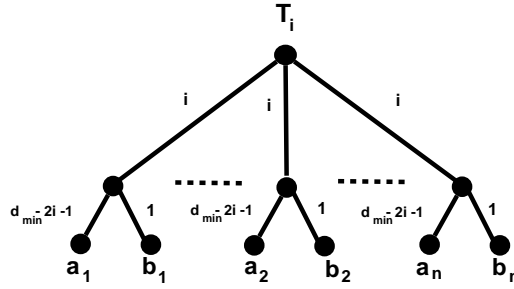


Fig. 6: The pairwise compatibility tree for the split antimatching graph G_i .

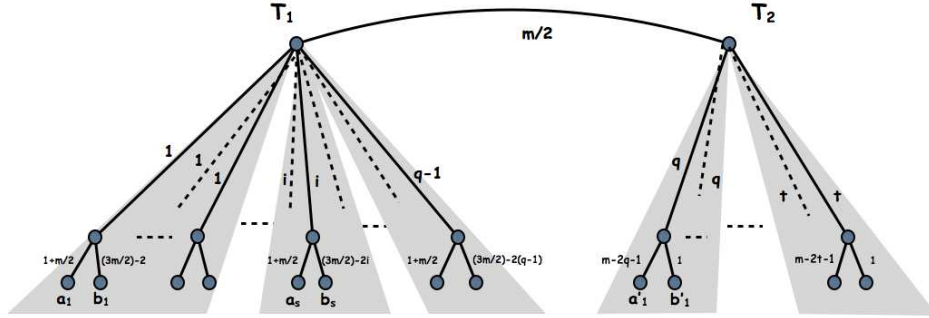


Fig. 7: The pairwise compatibility tree for the split matrogenic graph H .

Theorem 6 Let $H = G_1 \circ \dots \circ G_t$ be a split matrogenic graph such that for any split matching graph G_i and for any split antimatching graph G_j it holds that $i < j$. Then $H \in PCG$ and it is polynomial to find the tree and the values d_{min}, d_{max} associated to H .

Proof. Let $H = G_1 \circ \dots \circ G_t$. It is clear that if none of the graphs G_i is a split matching (a split antimatching) the proof trivially follows from Theorem 4 (Theorem 5). Hence, let G_q , $1 < q \leq t$, be the first occurrence of a split matching graph. Then, the graphs $H_1 = G_1 \circ \dots \circ G_{q-1}$ and $H_2 = G_q \circ \dots \circ G_t$ are a split matching sequence and a split antimatching sequence, respectively. Then, let $H_1 = LPG(T_1, M)$ where the tree is constructed in the same way as in the proof of Theorem 4 and $M = 2(t+1) + 1$ (recall that in the proof of Theorem 4 we only need M to be a value greater than $2q$). Similarly, according to the Theorem 5, $H_2 = mLPG(T_2, m)$ and $m = 2(t+1) + 1$ (note that we choose to have $m = M$). We modify T_2 in such a way that of the weights of the edges outcoming from the root start from value q instead of from 1 and the

other edges are modified accordingly. This is not restrictive, as T_2 results as if H_2 was the composition of t split antimatching graphs whose the first $q - 1$ are empty graphs.

We construct the pairwise compatibility tree T by joining the roots of T_1 and T_2 with an edge of weight $m/2$. We set $d_{min} = m$ and $d_{max} = 2m$. We modify the weights of the resulting tree increasing by $m/2$ the weight of any edge incident to a leaf in T_1 . Observe that in this way the distance of any two leaves in T_1 is increased by m . This means that two leaves correspond to vertices of an edge in H_1 if and only if their distance is less than or equal to $M + m = 2m$. Furthermore, the maximum distance of any two leaves in T_2 is less than or equal to $2m - 2t < 2m$ meaning that they correspond to vertices of an edge in H_2 if and only if their distance is greater than or equal to m .

We claim that $H = PCG(T, 2m, m)$ (recall that $m = 2(t + 1) + 1$). We have already shown that the pairwise compatibility constraints hold for any two leaves that correspond to two vertices of the same graph H_1 or H_2 . It remains to show it also holds for two leaves where one corresponds to a vertex in H_1 and the other to H_2 . To this purpose, let a_i and b_i be two distinct leaves in T_1 , connected to the root with edges of weight i and corresponding to vertices of the clique and the stable graph of H_1 , respectively. Similarly let a'_j, b'_j be two distinct leaves in T_2 , connected to the root with edges of weight j and corresponding to vertices in the clique and the stable graph of H_2 , respectively. The following hold:

- a) $d_T(a_i, a'_j) = 2m + i - j$ and as $i < j$ and $m > j$ then $m \leq 2m + i - j \leq 2m$. Hence, the corresponding vertices of a_i, a'_j in H are connected.
- b) $d_T(a_i, b'_j) = m + 1 + i + j + 1$ and as $m = 2t + 3 \geq i + j + 2$ then $m \leq m + i + j + 2 \leq 2m$. Hence, the corresponding vertices of a_i, b'_j in H are connected.
- c) $d_T(b_i, a'_j) = 2m - i + m - j - 1$ and as $m = 2t + 3 \geq i + j + 2$ then $2m + (m - i - j - 1) > 2m$. Hence, the corresponding vertices of b_i, a'_j in H are not connected.
- d) $d_T(b_i, b'_j) = 2m - i + j + 1$ and as $i < j$ then $2m + (i - j + 1) > 2m$. Hence, the corresponding vertices of b_i, b'_j in H are not connected.

This, concludes the proof. \square

It seems that the order of appearance of a matching or an antimatching sequence in a split matrogenic graph is somehow strictly related to the pairwise compatibility property. Indeed, in spite our efforts, the following problem remains open.

Problem: Let $H = G_1 \circ \dots \circ G_t$ be a split matrogenic graph such that for any split antimatching graph G_i and for any split matching graph G_j it holds that $i < j$. Is H a PCG ?

If this problem has an affirmative answer then it should not be difficult to prove that all split matrogenic graphs are PCG. Otherwise, the separation between the split matrogenic graphs that are PCG and those that are not, would be perfectly known.

5 Conclusions and Open Problems

In this paper we analyze the class of PCG with particular attention to two particular subclasses resulting when the pairwise compatibility constraints are relaxed. Hence, we consider the subclasses LPG and mLPG, resulting from the cases where $d_{min} = 0$ and $d_{max} = +\infty$, respectively. We study the relations between the classes PCG, LPG and mLPG. In particular, we show that the union of LPG and mLPG does not coincide with the whole class PCG, their intersection is not empty, and that neither of the classes LPG and mLPG is contained in the other. The graphs considered in these considerations are particular cases of the more general class of split matrogenic graphs. Hence, we attempt to determine whether the class of split matrogenic graphs belongs to PCG class. We prove that many split matrogenic graphs are PCG. However, the membership to PCG class of one particular split matrogenic graph remains an open problem.

It should be stressed that up to date, the pairwise compatibility has been investigated only for a few classes of graphs, thus determine this property for many graph classes remains an open problem. It is worth to mention that in [2] is shown that the clique problem can be solved in polynomial time for the class of compatibility graphs if we are able to construct in polynomial time a weighed tree that generates the pairwise compatibility graph. In view of this, it seems even more interesting to completely identify the Pairwise Compatibility graphs class.

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